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An integrable $sl(2|1)$ vertex model for the spin quantum Hall critical point

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Abstract. An integrable vertex model associated to the Lie superalgebra $sl(2|1)$ is constructed for the description of the spin quantum Hall critical phase. The model involves R-matrix solutions of the Yang–Baxter equation with respect to both the vector representation of $sl(2|1)$ and its dual and an inhomogenity in the spectral parameters. On the torus the model can be mapped onto a Chalker–Coddington-type network model.

1. Introduction

Recently, the problem of noninteracting quasiparticles in a disordered superconductor has attracted attention. Effective field theory descriptions of various cases have been derived from BCS mean-field Hamiltonians [1–4]. The quasiparticle Hamiltonian is invariant under $SU(2)$ spin rotations. For an inhomogeneous superconductor, time-reversal invariance may be broken by the presence of a magnetic field. The field theory is given by a chiral model associated to the Lie super algebra $osp(2n|2n)$ or by a nonlinear sigma model related to the symmetric space $osp(2n|2n)/gl(n|n)$ for preserved or broken time-reversal invariance, respectively. The second type of model is relevant to quasiparticles in the core of an isolated vortex in a disordered s-wave superconductor [1] as well as to quasiparticles in a dirty $d_{x^2-y^2}$ superconductor with an orbital coupling to a magnetic field [2, 5]. The coupling constant of the field theory represents the spin conductivity. Its evolution with the length scale of the system given directly by the system size or an inelastic scattering length due to a finite temperature is encoded by the beta-function of the field theory [6]. For two-dimensional systems described by the chiral model, results from renormalization group studies of nonlinear sigma models [7] indicate complete localization of all quasiparticles. However, extended states may arise in the case of the $osp(2n|2n)/gl(n|n)$ model. The corresponding weak localization effects have been investigated in [8].

In [2, 9], the analogy of the latter situation with the delocalization transition occurring in integer quantum Hall systems has been emphasized. As in the field theoretic formulation of the integer quantum Hall plateau transition [10], a topological term occurs in the action of the nonlinear sigma model. Its coupling constant is interpreted as the quantized spin Hall conductance characterizing each localized phase. In view of the similarities, the network model [11] developed to study the integer quantum Hall transition has been generalized for application in the present context [9]. The original Chalker–Coddington model yields a semiclassical description of a two-dimensional system of disordered electrons in a strong perpendicular magnetic field. It considers the guiding centre motion of spin-polarized electrons along the equipotential lines of a smoothly varying disorder potential. Quantum mechanical effects

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are taken into account allowing for scattering between different components of equipotentials referring to the same value in the vicinity of saddle points. For simplification, these scattering points are placed on the nodes of a regular lattice. Disorder is realized by random phases acquired by the electrons drifting along the links between the scattering nodes. The generalized model includes the spin degree of freedom. This is achieved by substituting the random $U(1)$ phases by random $U(2)$ matrices. The scattering at each node is characterized by a transfer matrix respecting spin reversal symmetry. A numerical investigation of this network model seems to reveal the existence of a critical point separating two insulating phases with a change of the quantized Hall conductance by two units. This conclusion is supported by a numerical study of a description in terms of a supersymmetric spin chain [12]. Using the density matrix renormalization group various universal critical properties such as the critical exponents related to the localization length and the density of states as well as the dimerization exponent have been obtained.

In this paper, an integrable vertex model is proposed for the description of the network model. Making use of the isomorphism between the Lie super algebras $osp(2|2)$ and $sl(2|1)$ the representation underlying the spin chain description and the network model is identified with the vector representation V of $sl(2|1)$ associated to a simple root system with one odd and one even simple root together with its dual representation V^*. The Boltzmann weights of the vertex model are given by the solution of the intertwining condition for the R-matrices with respect to the tensor products $V \otimes V$, $V^* \otimes V^*$, $V \otimes V^*$ and $V^* \otimes V$. Furthermore, an inhomogenity with respect to the spectral parameters of the R-matrices is incorporated. Both the vertex model and the network model are considered on the torus. The correlation functions of the network model can be extracted from those of the constructed integrable model.

After this work was completed, [12, 13] appeared at the preprint server. In [13] a mapping onto a percolation problem is used to determine critical exponents.

The paper is organized as follows. In section 2 definitions of the infinite-dimensional algebra underlying the integrable structure of the vertex model are given. Sections 2 and 3 consider the affine quantum superalgebra $U_q(\widehat{sl}(2|1))$ from which the integrable vertex model related to the network model is obtained in the limit $q \to 1$. The various R-matrices are given in section 3. The integrable diagonal vertex model is introduced in section 4 and the relation to the Chalker–Coddington-type model is pointed out in section 5.

2. $U_q(\widehat{sl}(2|1))$

Models based on the quantum super algebra $U_q(sl(2|1))$ have been investigated intensively for their relevance to one-dimensional interacting electronic systems (see, for example, [15] and references therein). In general, several systems of simple roots exist for a given Lie super algebra [18]. The nonlinear sigma model related to the problems studied in [1, 9, 12] is expressed in terms of a matrix field taking values in the quotient $OSp(2n|2n)/GL(n|n)$. A model with $n = 1$ is sufficient to capture the two-point spin conductance. Correlators involving more points require consideration of higher values of n. Making use of coherent states [14], the nonlinear sigma model may be realized as the continuum limit of a lattice model of Chalker–Coddington type (see section 5). To each link of this lattice model, a set of possible states is attributed. For $n = 1$, each set contains two bosonic and one fermionic state. The sets form representation spaces of the Lie super algebra $osp(2|2)$ realized by bilinears of two fermionic and one bosonic oscillator. The action of the $osp(2|2)$ generators on these representation spaces determines the Boltzmann weights of an anisotropic version of the lattice model. Thus, the model may be viewed as a construction based on the Lie superalgebra $sl(2|1)$ which is related to $osp(2|2)$ by an isomorphism. Then the three-dimensional representation spaces correspond to the vector representation of $sl(2|1)$ or to its dual representation, where the simple root system of $sl(2|1)$ contains one odd and one even root. The Cartan matrix for the corresponding affine Lie super algebra $\widehat{sl}(2|1)$ is given by

$$a = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & -2 \end{pmatrix}. \tag{1}$$

$U'_q(\widehat{sl}(2|1))$ is defined as the unital $\boldsymbol{Z}$-graded associative algebra generated by $\{e_n, f_n, q^{\pm h_n}, n = 0, 1, 2\}$ subject to the relations

$$\begin{aligned} &q^{h_n} q^{h_{n'}} = q^{h_{n'}} q^{h_n} \\ &q^{h_n} e_{n'} q^{-h_n} = q^{a_{nn'}} e_{n'} \\ &q^{h_n} f_{n'} q^{-h_n} = q^{-a_{nn'}} f_{n'} \\ &[e_n, f_{n'}] = \delta_{n,n'} \frac{q^{h_n} - q^{-h_n}}{q - q^{-1}} \end{aligned} \tag{2}$$

and [19]

$$\begin{aligned} &[e_1, e_2]_q e_2 - q^{-1} e_2 [e_1, e_2]_q = 0 \qquad [f_1, f_2]_{q^{-1}} f_2 - q f_2 [f_1, f_2]_{q^{-1}} = 0 \\ &[e_0, e_2]_q e_2 - q^{-1} e_2 [e_0, e_2]_q = 0 \qquad [f_0, f_2]_{q^{-1}} f_2 - q f_2 [f_0, f_2]_{q^{-1}} = 0 \end{aligned} \tag{3}$$

$$\begin{aligned} &[[e_0, e_1]_q, [e_0, e_2]_q]_q = 0 \qquad [[f_0, f_1]_{q^{-1}}, [f_0, f_2]_{q^{-1}}]_{q^{-1}} = 0 \\ &[[e_1, e_0]_q, [e_1, e_2]_q]_q = 0 \qquad [[f_1, f_0]_{q^{-1}}, [f_1, f_2]_{q^{-1}}]_{q^{-1}} = 0. \end{aligned} \tag{4}$$

In (2) $a_{nn'}$ denote the elements of the Cartan matrix (1). [,] is the usual Lie super bracket $[x, y] = xy - (-1)^{|x|\cdot|y|} yx$. The Serre relations (4) contain q-deformed super commutators defined by

$$\begin{aligned} &[e_n, e_{n'}]_q = e_n e_{n'} - (-1)^{|e_n|\cdot|e_{n'}|} q^{a_{nn'}} e_{n'} e_n \\ &[f_n, f_{n'}]_{q^{-1}} = f_n f_{n'} - (-1)^{|f_n|\cdot|f_{n'}|} q^{-a_{nn'}} f_{n'} f_n. \end{aligned} \tag{5}$$

The $\boldsymbol{Z}_2$-grading $|\cdot| : U_q(\widehat{sl}(2|1)) \to \boldsymbol{Z}_2$ is given by $|e_0| = |e_1| = |f_0| = |f_1| = 1$, $|e_2| = |f_2| = 0$ and $|q^{\pm h_n}| = 0\ \forall n$. Supplementing $U'_q(\widehat{sl}(2|1))$ by a grading operator d with commutators

$$[d, e_n] = \delta_{n,0} e_n \qquad [d, f_n] = -\delta_{n,0} f_n \qquad [d, d] = [d, q^{\pm h_n}] = 0 \tag{6}$$

yields the affine quantum superalgebra $U_q(\widehat{sl}(2|1))$. $U'_q(\widehat{sl}(2|1))$ admits a comultiplication

$$\Delta(e_n) = q^{h_n} \otimes e_n + e_n \otimes 1 \qquad \Delta(f_n) = f_n \otimes q^{-h_n} + 1 \otimes f_n \qquad \Delta(q^{\pm h_n}) = q^{\pm h_n} \otimes q^{\pm h_n} \tag{7}$$

and an antipode

$$S(e_n) = -q^{-h_n} e_n \qquad S(f_n) = -f_n q^{h_n} \qquad S(q^{\pm h_n}) = q^{\mp h_n}. \tag{8}$$

3. The R-matrices

Most studies of integrable models related to $U_q(\widehat{sl}(2|1))$ deal with constructions based on the vector representation of $U_q(sl(2|1))$. Guided by the structure of the nonlinear sigma models and the network model, a three-dimensional module $V = (v_1, v_2, v_3)$ is introduced with the $\boldsymbol{Z}_2$-grading $|v_0| = |v_1| = 0$ and $|v_2| = 1$. On the evaluation module $V_z = V \otimes \boldsymbol{C}[z, z^{-1}]$ a

$U'_q(\widehat{sl}(2|1))$-structure is given by

$$\begin{aligned} &f_2(v_0 \otimes z^n) = v_1 \otimes z^n \qquad e_1(v_2 \otimes z^n) = -v_1 \otimes z^n \\ &f_1(v_1 \otimes z^n) = v_2 \otimes z^n \qquad e_2(v_1 \otimes z^n) = -v_0 \otimes z^n \\ &f_0(v_2 \otimes z^n) = v_0 \otimes z^{n-1} \qquad e_0(v_0 \otimes z^n) = v_2 \otimes z^{n+1} \end{aligned} \tag{9}$$

$$\begin{aligned} &h_1(v_0 \otimes z^n) = 0 \qquad h_2(v_0 \otimes z^n) = -v_0 \otimes z^n \qquad h_0(v_0 \otimes z^n) = v_0 \otimes z^n \\ &h_1(v_1 \otimes z^n) = -v_1 \otimes z^n \qquad h_2(v_1 \otimes z^n) = v_1 \otimes z^n \qquad h_0(v_1 \otimes z^n) = 0 \\ &h_1(v_2 \otimes z^n) = -v_2 \otimes z^n \qquad h_2(v_2 \otimes z^n) = 0 \qquad h_0(v_2 \otimes z^n) = v_2 \otimes z^n. \end{aligned} \tag{10}$$

The graded Yang–Baxter equation is satisfied by the R-matrix with matrix elements

$$\begin{aligned} &R^{00}_{00}(z) = R^{11}_{11}(z) = 1 \qquad R^{22}_{22}(z) = \frac{1-q^2z}{q^2-z} \\ &R^{ij}_{ij}(z) = \frac{q(1-z)}{q^2-z} \qquad \text{with} \quad i,j = 0,1,2 \quad i \neq j \\ &R^{ji}_{ij}(z) = \frac{q^2-1}{q^2-z} \qquad \text{for} \quad i > j \\ &R^{ji}_{ij}(z) = \frac{z(q^2-1)}{q^2-z} \qquad \text{for} \quad i < j \\ &R^{kl}_{ij}(z) = 0 \qquad \text{for} \quad (k,l) \neq (i,j) \quad \text{or} \quad (k,l) \neq (j,i). \end{aligned} \tag{11}$$

Equation (11) refers to matrix elements R^{kl}_{ij} defined by

$$R(z)(v_i \otimes v_j) = \sum_{k,l} R^{kl}_{ij}(z)(v_k \otimes v_l). \tag{12}$$

The construction of a vertex model in the following section also involves the dual module $V^* = (v^*_0, v^*_1, v^*_2)$ with a $U_q(sl(2|1))$-structure determined by

$$\langle a(v^*)|v\rangle = (-1)^{|a|\cdot|v^*|}\langle v^*|S(a)v\rangle \qquad a \in U_q(sl(2|1)) \tag{13}$$

and the canonical pairing $\langle v^*_i|v_j\rangle = \delta_{i,j}$.

The corresponding R-matrix $R_{V^*V^*}(z)$ is related to $R_{VV}(z)$ by

$$R^{k^*l^*}_{i^*j^*}(z) = R^{lk}_{ji}(z) \tag{14}$$

if the normalization $R^{0^*0^*}_{0^*0^*}(z) = 1$ is adopted. With an analogous choice of the overall normalizations, the nonvanishing matrix elements of the mixed R-matrices $R_{VV^*}(z)$ and $R_{V^*V}(z)$ are

$$\begin{aligned} &R^{00^*}_{00^*}(z) = R^{11^*}_{11^*}(z) = 1 \qquad R^{0^*0}_{0^*0}(z) = R^{1^*1}_{1^*1}(z) = 1 \\ &R^{22^*}_{22^*}(z) = \frac{q^{-2}(q^4-z)}{1-z} \qquad R^{2^*2}_{2^*2}(z) = \frac{q^2(1-q^{-2}z)}{1-q^2z} \\ &R^{ij^*}_{ij^*}(z) = \frac{q^{-1}(q^2-z)}{1-z} \qquad R^{i^*j}_{i^*j}(z) = \frac{q(1-z)}{1-q^2z} \qquad i \neq j \\ &R^{jj^*}_{ii^*}(z) = -(-1)^{|i|}\frac{(1-q^{-2})q^{2j}}{1-z} \qquad R^{j^*j}_{i^*i}(z) = -\frac{z(q^2-1)}{1-q^2z} \qquad i > j \\ &R^{jj^*}_{ii^*}(z) = -\frac{z(q^2-1)q^{-2i}}{1-z} \qquad R^{j^*j}_{i^*i}(z) = -(-1)^{|j|}\frac{q^2-1}{1-q^2z} \qquad i < j. \end{aligned} \tag{15}$$

The R-matrix elements given in (11), (14) and (15) obey the Yang–Baxter equation on any tensor product of three modules chosen in $\{V, V^*\}$:

$$\begin{aligned} &R_{W_1W_2}(z)R_{W_1W_3}(zw)R_{W_2W_3}(w) = R_{W_2W_3}(w)R_{W_1W_3}(zw)R_{W_1W_2}(z) \\ &W_i \in \{V, V^*\} \quad i = 1,2,3. \end{aligned} \tag{16}$$

On each tensor product of two modules V or V^*, the unitarity condition is satisfied:

$$\begin{aligned}\sum_{k,l}(-1)^{|k|\cdot|l|}R^{nm}_{lk}(z^{-1})R^{kl}_{ij}(z) &= \sum_{k,l}(-1)^{|k|\cdot|l|}R^{n^*m^*}_{l^*k^*}(z^{-1})R^{k^*l^*}_{i^*j^*}(z) = (-1)^{|i|\cdot|j|}\delta_{i,m}\delta_{j,n}\\ \sum_{k,l}(-1)^{|k|\cdot|l|}R^{n^*m}_{l^*k}(z^{-1})R^{kl^*}_{ij^*}(z) &= \sum_{k,l}(-1)^{|k|\cdot|l|}R^{nm^*}_{lk^*}(z^{-1})R^{k^*l}_{i^*j}(z) = (-1)^{|i|\cdot|j|}\delta_{i,m}\delta_{j,n}.\end{aligned} \tag{17}$$

Integrable models related to finite-dimensional representations of $sl(n|m)$ have been considered quite extensively ([15–17] and references in [20, 21]) in context with link polynomials [22] and with one-dimensional systems of interacting electrons [23]. R-matrices related to the vector representation of $U_q(sl(n|m))$ have first appeared implicitly in solutions of the Yang–Baxter equation for a particular nested model [24]. Later, rational and trigonometric solutions of the Yang–Baxter equation associated to $gl(n|m)$ have been studied (see [20] for references).

4. The diagonal vertex model

Boltzmann weights $\{\hat{R}^{kl}_{ij}(z)\}$ of an integrable vertex model follow from solutions $\{R^{kl}_{ij}(z)\}$ of the graded Young–Baxter equation by the transformation

$$\hat{R}^{kl}_{ij}(z) = (-1)^{|k|\cdot|l|}R^{kl}_{ij}(z). \tag{18}$$

In the remainder, the operator $\bar{R}(z) = P^{gr}R(z)$ will also be used. The graded permutation operator P^{gr} is defined by $P^{gr}(w_1 \otimes w_2) = (-1)^{|w_1|\cdot|w_2|}w_2 \otimes w_1$.

In this section, vertex models are constructed from the rational limits of the R-matrices given in the preceding section. Expressions (11), (14) and (15) may be rewritten with the replacements $q = \mathrm{e}^{\epsilon}$ and $z = \mathrm{e}^{-2\epsilon u}$. In the limit $\epsilon \to 0$ the Lie-superalgebra symmetry of the R-matrix is restored. From the $U_q(\widehat{sl}(2|1))$ R-matrices one obtains

$$\begin{aligned}R^{kl}_{ij}(u) &= R^{k^*l^*}_{i^*j^*}(u) = \frac{u}{u+1}\delta_{i,k}\delta_{j,l} + (-1)^{|i|\cdot|j|}\frac{1}{u+1}\delta_{i,l}\delta_{j,k}\\ R^{kl^*}_{ij^*}(u) &= \frac{u}{u-1}\delta_{i,k}\delta_{j,l} - (-1)^{|k|}\frac{1}{u-1}\delta_{i,j}\delta_{k,l}\\ R^{k^*l}_{i^*j}(u) &= \frac{u+1}{u}\delta_{i,k}\delta_{j,l} - (-1)^{|i|}\frac{1}{u}\delta_{i,j}\delta_{k,l}.\end{aligned} \tag{19}$$

In the following the R-matrices will also be expressed in terms of the graded permutation and monoid operators or the quadratic Casimir C of $sl(2|1)$:

$$\begin{aligned}R_{VV} &= R_{V^*V^*} = \frac{u}{u+1}\left(\mathrm{id} + \frac{1}{u}P^{gr}\right) = \mathrm{id} - \frac{1}{u+1}\Delta(C)\\ R_{VV^*} &= \frac{u}{u-1}\left(\mathrm{id} - \frac{1}{u}O^{gr}\right) = \mathrm{id} - \frac{1}{u-1}\Delta(C)\\ R_{V^*V} &= \frac{u+1}{u}\left(\mathrm{id} - \frac{1}{u+1}O^{gr}\right) = \mathrm{id} - \frac{1}{u}\Delta(C).\end{aligned} \tag{20}$$

According to (19) the action of the graded monoid operator O^{gr} on $V \otimes V^*$ and $V^* \otimes V$ is given by $O^{gr}(v_1 \otimes v_j^*) = \delta_{i,j}\sum_{k=0,1,2}(-1)^{|k|}v_k \otimes v_k^*$ and $O^{gr}(v_i^* \otimes v_j) = \delta_{i,j}(-1)^{|i|}\sum_{k=0,1,2}v_k^* \otimes v_k$.

To each link of a regular diagonal lattice (see figure 1) a statistical variable is associated taking either values in V or V^*. The modules V and V^* are assigned to the lines according to the alternating sequence

$$\cdots \otimes V \otimes V^* \otimes V \otimes V^* \otimes V \otimes V^* \otimes V \otimes \tag{21}$$

on the diagonal lines pointing from bottom left to top right and according to

$$\cdots \otimes V \otimes V \otimes V^* \otimes V^* \otimes V \otimes V \otimes V^* \otimes V^* \otimes V \otimes V \otimes \cdots \tag{22}$$

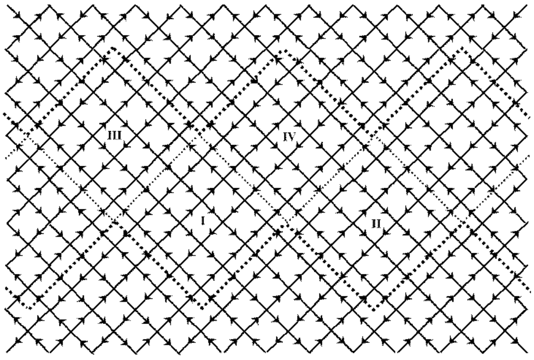

Figure 1. The diagonal vertex model. The evolution operator is represented by the section between the thick dotted lines. Arrows pointing to northwest or northeast distinguish links with modules V from those with modules V^* (arrows pointing southwest or southeast).

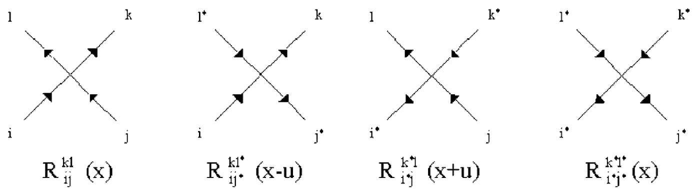

Figure 2. The R-matrices.

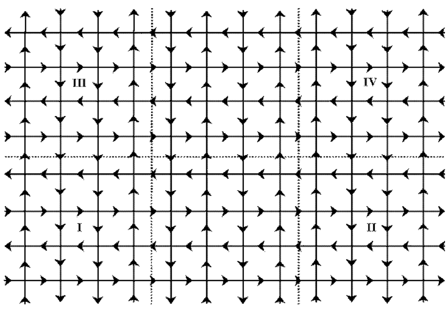

Figure 3. The transfer matrix $T^{inh}(x, u)$. Arrows pointing up or right represent links with the module V and the remaining ones links with the module V^*.

on the other diagonals. To the crossings, Boltzmann weights (18), (19) varying with two parameters x and u are associated. The assignment of the indices as well as the dependence on x and u are indicated in figure 2. The model is considered for small values of x and large values of u. Thus expansions in x and $1/u$ are expected to be appropriate. On this vertex model, a particular evolution operator $T(x, u)$ may be introduced. Its graphical representation is provided by the section of the lattice within the thick dotted lines shown in figure 1. Given periodic boundary condition in the horizontal direction, the evolution operator can be related to the row-to-row transfer matrix of the corresponding vertex model with horizontal and vertical links (see figure 3). The latter has modules V and V^* associated to its horizontal links following sequence (21). An additional inhomogenity is allowed for by dividing the lattice into vertical strips each of them including four vertical lines. To the vertical lines of every second strip the modules V and V^* are assigned to according to the sequence $V \otimes V^* \otimes V^* \otimes V$. In these strips the assignment of indices and arguments of the Boltzmann weights to the links is obtained from the one shown in figure 2 by means of a clockwise rotation by $\pi/4$. To the vertical lines in the remaining strips the modules are associated according to the sequence $V^* \otimes V \otimes V^* \otimes V$. Within these strips the assignment of R-matrices indicated in figure 2 applies with $x = 0$. Imposing again periodic boundary conditions in the horizontal direction the transfer matrix $T^{inh}(x, u)$: $(\otimes V \otimes V^* \otimes V^* \otimes V \otimes V^* \otimes V \otimes V^* \otimes V)^N \to (\otimes V \otimes V^* \otimes V^* \otimes V \otimes V^* \otimes V \otimes V^* \otimes V)^N$ is represented by a horizontal section of the lattice model including four neighbouring horizontal lines as shown in figure 3. This transfer matrix $T^{inh}(x, u)$ may be decomposed into transfer matrices $T^{(a)}(x, u)$, $a = 1, 2$:

$$T^{inh}(x, u) = T^{(2)}(x, u)T^{(1)}(x, u)T^{(2)}(x, u)T^{(1)}(x, u). \tag{23}$$

Each transfer matrix $T^{(a)}(x, u)$ includes a single horizontal line with the auxiliary space V (V^*) for $a = 1$ ($a = 2$). The rational limit of the Yang–Baxter equations (16) implies the commutation of any two transfer matrices $T^{(a)}(x, u)$, $T^{(b)}(x', u')$, $a, b = 1, 2$. Thus two composite transfer matrices $T^{inh}(x, u)$ and $T^{inh}(x', u')$ commute. Due to the initial condition $R_{i\,j}^{k\,l}(0) = R_{i^*j^*}^{k^*l^*}(0) = \delta_{i,l}\,\delta_{j,k}$ and the rational limit of the unitarity property (17), the action of the vertical strips with $x = 0$ amounts to a diagonal shift in the southwest–northeast direction. Consequently, the action of the product $(T^{inh}(x, u))^2$ is equivalent to the action of the evolution operator $T(x, u)$ of the diagonal lattice model. In figures 1 and 3, the quadratic regions I–IV enclosed by the dotted lines indicate corresponding sections of $T(x, u)$ and $T^{inh}(x, u)$. The reasoning given above is a generalization of the argument presented in [25] for a homogeneous eight-vertex model.

The integrable vertex model proposed in this section may be studied by means of the algebraic Bethe ansatz. Since the transfer matrices $T^{(a)}(x, u)$ commute, the spectrum of $T^{inh}(x, u)$ can be obtained by diagonalizing $T^{(1)}(x, u)$ and $T^{(2)}(x, u)$ separately. The corresponding monodromy matrices $T_i^{(a)j}(x, u)$ satisfy the commutation relations

$$\begin{aligned}
&\sum_{k,l} R_{kl}^{mn}(x)T_i^{(1)k}(xx', u)T_j^{(1)l}(x', u) = \sum_{k,l} T_l^{(1)n}(x', u)T_k^{(1)m}(xx', u)R_{ij}^{kl}(x)\\
&\sum_{k,l} R_{k^*l^*}^{m^*n^*}(x)T_{i^*}^{(2)k^*}(xx', u)T_{j^*}^{(2)l^*}(x', u) = \sum_{k,l} T_{l^*}^{(2)n^*}(x', u)T_{k^*}^{(2)m^*}(xx', u)R_{i^*j^*}^{k^*l^*}(x).
\end{aligned} \tag{24}$$

Starting from the *RTT*-equations (24) the algebraic Bethe ansatz procedure outlined in [26] may be adopted. By now, a variety of vertex models or spin chains based on representations of the Lie superalgebras $sl(2|1)$ and $gl(2|1)$ have been investigated by means of this method (see [16, 17, 27] and references given there). In particular, the algebraic Bethe ansatz has been used in a study of another staggered vertex model involving the vector representation of $sl(2|1)$ and an inhomogeneity in the spectral parameter [17].

5. The relation to the network model

Returning to the diagonal lattice model, let us denote the pair of horizontal and vertical coordinates of a link by $\boldsymbol{m} = (m_1, m_2)$. Two links with positions $\boldsymbol{m}$ and $\boldsymbol{n}$ may be chosen on the lattice such that the module V is assigned to the link at $\boldsymbol{m}$ and its dual module V^* to the link at $\boldsymbol{n}$. Then the two-point function $P_{\boldsymbol{m},\boldsymbol{n}}^{i,j^*}(x, u)$ is introduced as the probability that the statistical variables $l_{\boldsymbol{m}}$ and $l_{\boldsymbol{n}}$ take the values v_i and v_j^*, respectively:

$$P_{\boldsymbol{m},\boldsymbol{n}}^{i,j^*}(x, u) = \sum_{\{l_k\}} \delta_{l_{\boldsymbol{m}}, v_i}\delta_{l_{\boldsymbol{n}}, v_j^*} \prod_{t_1=0}^{8N-1} \prod_{t_2=0}^{8M-1} \bar{R}_{l(t_1,t_2),l(t_1+1,t_2)}^{l(t_1,t_2+1),l(t_1+1,t_2+1)}(x, u). \tag{25}$$

The right-hand side has been written in terms of the matrix elements of $\bar{R}(u) = P^{gr}R(u)$. To facilitate notation, the indices of the latter do not distinguish between links with module V and those with V^*. The partition function is normalized to one for the periodic boundary condition in the vertical direction. In this case, the introduction of a modified evolution operator

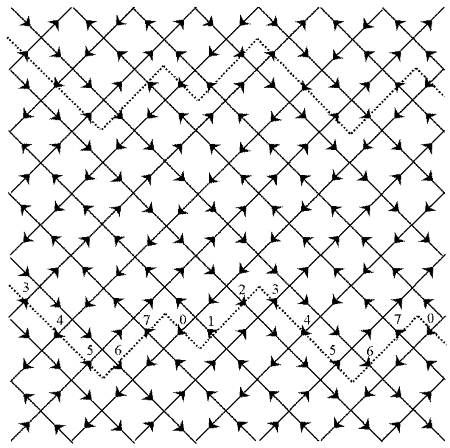

Figure 4. The modified evolution operator of the vertex model.

$\tilde{T}(x,u)$ for the diagonal network model turns out to be useful. In the pictorial representation it corresponds to the section between the dotted lines in figure 4. $\tilde{T}(x,u)$ acts on the space of states $(\otimes V\otimes V^*)^{4N}$. Within a block $(\otimes V\otimes V^*)^4$ the links may be labelled from 0 to 7 as indicated in figure 4.

For the $sl(2|1)$-model one finds from (19)

$$\lim_{x\to 0}\bar{R}_{VV}(x) = \lim_{x\to 0}\bar{R}_{V^*V^*}(x) = \mathrm{id} \tag{26}$$

$$\lim_{u\to\pm\infty}\bar{R}_{V^*V}(u) = \lim_{u\to\pm\infty}\bar{R}_{VV^*}(u) = P_{gr}. \tag{27}$$

Thus in the limit $x\to 0$, $u\to\infty$ $\tilde{T}(x,u)$ reduces to the identity operator. An expansion in x and $1/u$ yields

$$\begin{aligned}\ln\tilde{T}(x,u) = {} & -(16N\ln(1+x)+8N\ln(u+x)(u-x-1)-8Nx^2+\mathrm{O}(x^3))\mathrm{id}\\ & +(2x-2x^2+\mathrm{O}(x^3))\sum_{n=0}^{8N-1}P^{gr}_{n,n+2}\\ & +\left(\frac{2x}{(u+x)(u-x-1)}+\frac{2x^2}{(u+x)^2(u-x-1)^2}+\mathrm{O}\left(\frac{x^3}{u^6}\right)\right)\sum_{n=0}^{8N-1}O^{gr}_{n,n+1}\\ & +U(x,u)\end{aligned} \tag{28}$$

where the operators $O^{gr}_{n,n+1}$ on W_N act on $V_n\otimes V^*_{n+1}$ or $V^*_n\otimes V_{n+1}$ as the monoid operator and as the identity operator on the remaining part of W_N. Similarly, $P^{gr}_{n,n+2}$ acts on $V_n\otimes V^*_{n+1}\otimes V_{n+2}$ or $V^*_n\otimes V_{n+1}\otimes V^*_{n+2}$ according to $P^{gr}_{13}(a\otimes b\otimes c) = (-1)^{(|a|\cdot|b|+|a|\cdot|c|+|b|\cdot|c|)}\, c\otimes b\otimes a$ and as the identity elsewhere. The last term $U(x,u)$ contains only commutators of P^{gr} and O^{gr}:

$$\begin{aligned}U(x,u) = {} & -2\frac{x}{u+x}\sum_{k=0}^{N-1}([P_k^{(0,2)},O_k^{(0,1)}]+[P_k^{(6,8)},O_k^{(6,7)}]+[P_k^{(7,9)},O_k^{(8,9)}])+2\frac{x}{u-x-1}\\ & \times\sum_{k=0}^{N-1}([P_k^{(1,3)},O_k^{(1,2)}]+[P_k^{(2,4)}+P_k^{(3,5)},O_k^{(3,4)}]+[P_k^{(4,6)}+P_k^{(5,7)},O_k^{(5,6)}])\\ & -\left(\frac{x}{u+x}-\frac{x}{u-x-1}\right)\sum_{k=0}^{N-1}([P_k^{(1,3)},O_k^{(0,1)}+O_k^{(3,4)}]\end{aligned}$$

$$
\begin{aligned}
&+[P_k^{(3,5)}, O_k^{(2,3)} - O_k^{(5,6)}] - [P_k^{5,7)}, O_k^{(4,5)} + O_k^{(7,8)}] \\
&+[P_k^{(6,8)}, O_k^{(5,6)} + O_k^{(8,9)}] - [P_k^{(8,10)}, O_k^{(7,8)} + O_k^{(10,11)}]) \\
&+\frac{1}{2}\left(\frac{1}{u+x} - \frac{1}{u-x-1}\right)^2 \sum_{k=0}^{N-1}([O_k^{(1,2)}, O_k^{(0,1)} - O_k^{(2,3)}] \\
&-[O_k^{(3,4)}, O_k^{(2,3)} + O_k^{(4,5)}] - [O_k^{(5,6)}, O_k^{(4,5)} + O_k^{(6,7)}] \\
&+[O_k^{(7,8)}, O_k^{(6,7)} + O_k^{(8,9)}]) + \mathrm{O}(x^2) + \mathrm{O}\left(\frac{x}{u^2}\right).
\end{aligned}
\tag{29}
$$

In (29) the graded permutation and monoid operators are abbreviated by $P_k^{(i,i+2)} \equiv P^{gr}_{8k+i,8k+i+2}$ and $O_k^{(i,i+1)} \equiv O^{gr}_{8k+i,8k+i+1}$. For $i > 7$, the positions $8(N-1)+i$ and i are identified. If periodic boundary conditions in the vertical direction are applied to the product $(\tilde{T}(x,u))^M$, the contributions from $U(x,u)$ cancel. Then the partition function is given by

$$
\begin{aligned}
Z = {} & ((1+x)^2(1+x^2)^{-1}(u+x)(u-x-1))^{-8NM} \\
& \times trg_{W_N}\left\{ \exp\left(M \sum_{n=0}^{8N-1} \ln\left(1 + \frac{2x}{1+x^2} P^{gr}_{n,n+2}\right)\right.\right. \\
& \left.\left. + M \ln\left(1 + \frac{2x}{(u+x)(u-x-1)}\right) \sum_{n=0}^{8N-1} O^{gr}_{n,n+1}\right)\right\}.
\end{aligned}
\tag{30}
$$

Here trg_{W_N} denotes the graded trace over the space of states $W_N = (\otimes V \otimes V^*)^{4N}$. The remainder of this section specializes to the case $N = pM$, $p \in \boldsymbol{N}$. In the nonrational case each term in the expansion of the partition sum is represented by a link [28] built from the braid, the monoid and the identity operators. For $i \neq j$, the two-point function $P^{i,j^*}_{\boldsymbol{m},\boldsymbol{n}}(x,u)$ selects all the links with the positions $\boldsymbol{m}, \boldsymbol{n}$ placed on different components of the link. In the rational limit the braid operator is substituted by the permutation operator. The set of links generated only from the identity and the monoid operators in (30) coincides with the links of a diagonal lattice model of Chalker–Coddington type [11]. In contrast to the lattice model construction described in the preceding section, all lattice links of a Chalker–Codington-type model with even horizontal position carry the module V while the dual module V^* is assigned to all links with odd horizontal position. Thus only two types of vertices arise. An anisotropic version of the model has Boltzmann weights

$$
W^{k^*l}_{i^*j}(y) = \delta_{i,k}\delta_{j,l} + (-1)^{|i|} y \delta_{i,j}\delta_{k,l} \qquad W^{kl^*}_{ij^*}(y) = \delta_{i,k}\delta_{j,l} + (-1)^{|k|} y \delta_{i,j}\delta_{k,l}. \tag{31}
$$

Here i, j, k, l refer to the lower left and right and to the upper left and right entry of each diagonal vertex, respectively. Periodic boundary conditions are applied in both directions. The model consists out of $2M$ raws of links. To relate horizontal and vertical positions of both lattice models, an evolution operator $T^{CC}(y)$ is introduced for the Chalker–Coddington-type model as indicated by the dotted lines in figure 5 where the lattice links with horizontal positions $\tilde{m}_1 = 8n + i$, $i = 0, 1, \ldots, 7$ are specified. For these links, the vertical position $\tilde{m}_2 = 0$ is introduced. The periodic boundary condition in the horizontal direction of the diagonal vertex model yields a periodic boundary condition with respect to the vertical position $\tilde{m}_2$. $T^{CC}(y)$ acts on the space $(\otimes V \otimes V^*)^{4N}$. The appropriate mapping $\sigma : X_{N,M} \to Y_{N,M}$ of a subset $X_{N,M}$ of lattice positions $\{m_1, m_2\}$ of the network model in section 4 to the set $Y_{N,M}$ of lattice

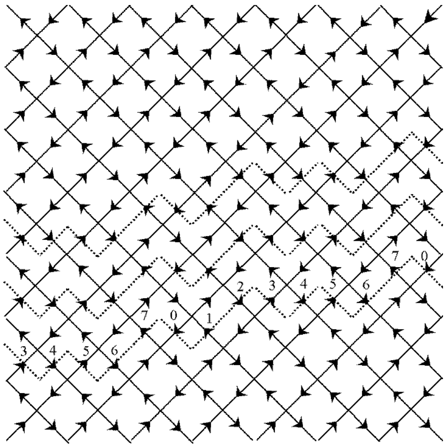

Figure 5. The network model.

positions $\{\tilde{m}_1, \tilde{m}_2\}$ of the Chalker–Coddington-type model is given by

$$\begin{aligned}
&\sigma(8n, 2l) = (8n, 2l) \qquad \sigma(8n+4, 2l+4) = (8n+4, 2l+1)\\
&\sigma(8n+1, 2l+3) = (8n+1, 2l+1) \qquad \sigma(8n+5, 2l) = (8n+5, 2l)\\
&\sigma(8n+2, 2l+5) = (8n+2, 2l+1) \qquad \sigma(8n+6, 2l+3) = (8n+6, 2l+1)\\
&\sigma(8n+3, 2l) = (8n+3, 2l) \qquad \sigma(8n+7, 2l+5) = (8n+7, 2l+1)
\end{aligned} \tag{32}$$

and

$$\begin{aligned}
&\sigma(8n, 2l+3) = \sigma(8n, 2l+4) = \sigma(8n, 2l+5) = (8n, 2l+1)\\
&\sigma(8n+1, 2l-2) = \sigma(8n+1, 2l-1) = \sigma(8n+1, 2l) = (8n+1, 2l)\\
&\sigma(8n+2, 2l) = \sigma(8n+2, 2l+1) = \sigma(8n+2, 2l+2) = (8n+2, 2l)\\
&\sigma(8n+3, 2l+3) = \sigma(8n+3, 2l+4) = \sigma(8n+3, 2l+5) = (8n+3, 2l+1)\\
&\sigma(8n+4, 2l-1) = \sigma(8n+4, 2l) = \sigma(8n+4, 2l+1) = (8n+4, 2l)\\
&\sigma(8n+5, 2l+3) = \sigma(8n+5, 2l+4) = \sigma(8n+5, 2l+5) = (8n+5, 2l+1)\\
&\sigma(8n+6, 2l-2) = \sigma(8n+6, 2l-1) = \sigma(8n+6, 2l) = (8n+6, 2l)\\
&\sigma(8n+7, 2l) = \sigma(8n+7, 2l+1) = \sigma(8n+7, 2l+2) = (8n+7, 2l)
\end{aligned} \tag{33}$$

with $n = 0, 1, \ldots, N-1$ and $l = 0, 1, \ldots, M-1$. In (33) the vertical positions $m_2 = -1$ and $m_2 = -2$ are identified with $m_2 = 8M-1$ and $m_2 = 8M-2$, respectively. A link composed out of l monoid operators from (30) and identity operators has a weight $v^l \equiv (\frac{2x}{(u+x)(u-x-1)})^l$. Thus the two-point function $\hat{P}^{i,j^*}_{\boldsymbol{m},\boldsymbol{n}}(y)$ obtained from $P^{i,j^*}_{\boldsymbol{m},\boldsymbol{n}}(x, v)$ after discarding all links with contributions of the permutation operators equals the two-point function $Q^{i,j^*}_{\sigma(\boldsymbol{m}),\sigma(\boldsymbol{n})}(y)$ of an anisotropic Chalker–Coddington model if $\boldsymbol{m}$ and $\boldsymbol{n}$ are chosen from the subset $X_{N,M}$:

$$\hat{P}^{i,j^*}_{\boldsymbol{m},\boldsymbol{n}}(y) = \exp\left(y\frac{\mathrm{d}}{\mathrm{d}v}\right) P^{i,j^*}_{\boldsymbol{m},\boldsymbol{n}}(x, v)\bigg|_{x=v=0} = Q^{i,j^*}_{\sigma(\boldsymbol{m}),\sigma(\boldsymbol{n})}(y) \qquad \text{for} \quad \boldsymbol{m}, \boldsymbol{n} \in X_{N,M}. \tag{34}$$

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